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DAMPED LATERAL VIBRATION IN AN AXIALLY CREEPING BEAM WITH RANDOM MATERIAL PARAMETERS†

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Abstract—Damped lateral vibration in an axially creeping beam with random material parameters is considered. The temperature and imperfection density, and as a result two inelastic material parameters, are random functions of the distance along the neutral axis. It is assumed that the material of the beam is governed by a nonlinear Maxwell model, and that the initial tensile stress is much greater than the increment of stress caused by the oscillation so that a perturbation technique may be employed. Two special cases are fully analyzed: (a) the random parameters are random processes which are only slightly random, and (b) the random parameters are random variables which are largely random. Statistical results are obtained for the lateral velocity, bending moment, logarithmic decrement and circular frequency. It is found that in a specific example, where the beam is made of an aluminum alloy, the lateral velocity, bending moment and logarithmic decrement are sensitive to randomness in temperature and imperfection density, whereas the circular frequency is almost deterministic.

1. INTRODUCTION

RECENTLY, the study of the dynamic characteristics of elastic beams having random stiffness parameters has received some attention (e.g., see Bliven and Soong [1]). However, randomness is considerably more pronounced in structure and temperature sensitive nonlinear inelastic material parameters, and this effect has been studied by Soong and Cozzarelli [2], Parkus [3], Parkus and Bargmann [4] and Cozzarelli and Huang [5]. As an extension of the work given in [5], the static problem of steady creep bending in a beam with random inelastic material parameters was considered in [6]. The study of the dynamic characteristics of nonlinear inelastic beams having random inelastic material parameters has received virtually no attention to date, and thus we shall now turn our attention to this problem.

The present analysis is concerned specifically with damped lateral vibration in an axially creeping beam, where the inelastic material parameters are described in a stochastic sense. It is assumed that the material of the beam is governed by a nonlinear Maxwell model, composed of a linear elastic element with a deterministic material stiffness and a nonlinear viscous element with two random parameters—a viscosity parameter $\dot{\varepsilon}_c$ and a creep power n.

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The method of approach employed here is analogous to that used in Cozzarelli, Wu and Tang [7], i.e., the initial tensile stress is assumed to be much greater than the increment of stress caused by the oscillation and as a result a perturbation technique may be employed. We then find that the nonlinear viscoelastic problem posed above is replaced by an equivalent linear viscoelastic problem. Also, we separate the problem into two uncoupled problems—the "random temperature problem" and the "random imperfection problem". In the random temperature problem only $\dot{\varepsilon}_c$ is random, whereas in the random imperfection problem only n is random. It was shown in [5] that the random inelastic material parameters $\dot{\varepsilon}_c$ and n have a similar statistical behavior, i.e., they are both lognormal. After the application of the perturbation technique, these two problems reduce to mathematically equivalent problems. Thus, we shall only present the details for the random temperature problem but will give the final results for both problems.

In this study it is assumed that the temperature and imperfection density and consequently the inelastic material parameters are random functions of the distance along the neutral axis of the beam. We are then faced with the problem of solving two simultaneous differential equations in velocity and bending moment containing a random process coefficient. The analysis of this problem is difficult, and thus we shall confine ourselves to two special cases: (a) the random parameters are random processes which are only slightly random, and (b) the random parameters are random variables which are largely random. In case (a) the perturbation method suggested by Keller [8] will be used to find the mean velocity and bending moment. In case (b) we employ standard procedures to determine the mean velocity and bending moment; the density function and statistical moments of the logarithmic decrement and circular frequency are also obtained.

The formulation of the governing equations with lognormal statistics for the inelastic material parameters is presented in Section 2. This is followed in Sections 3 and 4 by a discussion of the two separate cases mentioned in the preceding paragraph. In both cases a specific numerical example is considered, where the beam is made of aluminum alloy 1100. The results are summarized in the final section.

2. FORMULATION OF GOVERNING EQUATIONS

2.1. *Constitutive equation with random parameters*

For creeping metals the strain depends not only on time but also exhibits a nonlinear dependence on stress. In order to account for these effects, we shall employ the mathematically convenient as well as physically plausible nonlinear Maxwell model. This model consists of a linear elastic element with a material stiffness E in series with a nonlinear viscous element with two parameters—a creep parameter λ and a creep power n. Parameters λ and n are highly sensitive to moderate fluctuations in temperature and imperfection density, whereas E is almost insensitive to such fluctuations. Thus, it is assumed that these two inelastic material parameters are random functions of some space coordinate X whereas E is deterministic. Using a carat over a symbol to indicate a random quantity, the corresponding constitutive equation in the one-dimensional case will then be written as

$$\hat{\dot{\varepsilon}}_x(X, \bar{T}) = \frac{1}{E}\frac{\partial \hat{\sigma}_x(X, \bar{T})}{\partial \bar{T}} + \left|\frac{\hat{\sigma}_x(X, \bar{T})}{\hat{\lambda}(X)}\right|^{\hat{n}(X)} \operatorname{sgn}(\hat{\sigma}_x). \tag{1}$$

Here, the stress $\hat{\sigma}_x(X, \bar{T})$ and strain rate $\hat{\dot{\varepsilon}}_x(X, \bar{T})$ are random functions of X, the time $\bar{T}$ and possibly some other space coordinates, and $\mathrm{sgn}(\hat{\sigma}_x)$ is the signum function defined as

$$\mathrm{sgn}(\hat{\sigma}_x) = \begin{cases} 1 & \hat{\sigma}_x > 0 \\ -1 & \hat{\sigma}_x < 0. \end{cases} \tag{2}$$

Adopting the notation used in [6], we can rewrite equation (1) in a more convenient form as

$$\hat{\dot{\varepsilon}}_x(X, \bar{T}) = \frac{1}{E}\frac{\partial \hat{\sigma}_x(X, \bar{T})}{\partial \bar{T}} + \hat{\dot{\varepsilon}}_c(X)\left|\frac{\hat{\sigma}_x(X, \bar{T})}{\sigma_c}\right|^{\hat{n}(X)} \mathrm{sgn}(\hat{\sigma}_x). \tag{3}$$

Here,

$$\hat{\dot{\varepsilon}}_c(X) = \left(\frac{\sigma_c}{\hat{\lambda}(X)}\right)^{\hat{n}(X)} \tag{4}$$

is a random viscosity parameter equal to the random strain rate obtained when $\hat{\sigma}_x(X, \bar{T}) = \sigma_c$, an arbitrary deterministic constant reference stress.

In Section 2(b) we will consider a Maxwell material with an initial constant deterministic tensile stress σ_{xi}, subjected to a small lateral disturbance at $\bar{T} = 0$. This disturbance is translated into a small additional random increment of stress, i.e.

$$\hat{\sigma}_x(X, \bar{T}) = \sigma_{xi} + \varepsilon\tilde{\hat{\sigma}}_x(X, \bar{T}) \tag{5}$$

where $\tilde{\hat{\sigma}}_x(X, \bar{T})$ is a random function and ε is a small quantity. Also, as a result of this disturbance the strain rate $\hat{\dot{\varepsilon}}_x(X, \bar{T})$ assumes the form

$$\hat{\dot{\varepsilon}}_x(X, \bar{T}) = \hat{\dot{\varepsilon}}_{xi}(X) + \varepsilon\tilde{\hat{\dot{\varepsilon}}}_x(X, \bar{T}) + \ldots. \tag{6}$$

Substituting equations (5) and (6) into equation (3), subtracting the relation of the initial state, and grouping the terms of order ε, we obtain a simplified linear constitutive relation for this case as

$$\tilde{\hat{\dot{\varepsilon}}}_x(X, \bar{T}) = \frac{1}{E}\frac{\partial \tilde{\hat{\sigma}}_x(X, \bar{T})}{\partial \bar{T}} + \frac{1}{\sigma_{xi}}\hat{\dot{\varepsilon}}_c(X)\hat{n}(X)\tilde{\hat{\sigma}}_x(X, \bar{T}). \tag{7}$$

Here, use has been made of the relation $\sigma_c = \sigma_{xi}$.

For convenience, we summarize some of the statistical results pertaining to $\hat{\dot{\varepsilon}}_c(X)$ and $\hat{n}(X)$ obtained in [5]. It is assumed that $\hat{\dot{\varepsilon}}_c(X)$ is a function of the random temperature $\hat{T}(X)$ only, and that $\hat{T}(X)$ is a homogeneous normal random process in X with mean T_0 and variance $\sigma_{\hat{T}}^2$. We define a nondimensional viscosity parameter $\hat{\mathscr{E}}(X)$ and a nondimensional temperature $\hat{\tau}(X)$ as

$$\hat{\mathscr{E}}(X) = \frac{\hat{\dot{\varepsilon}}_c(X)}{\dot{\varepsilon}_{c0}} \tag{8a}$$

$$\hat{\tau}(X) = \frac{B[\hat{T}(X) - T_0]}{T_0^2} \tag{8b}$$

where $\dot{\varepsilon}_{c0}$ is the nominal value of $\hat{\dot{\varepsilon}}_c(X)$ (value at $\hat{T} = T_0$) and B is a deterministic creep constant. The parameter $\hat{\mathscr{E}}(X)$ is related to $\hat{\tau}(X)$ by the expression

$$\hat{\mathscr{E}}(X) = e^{\hat{\tau}(X)}. \tag{9}$$

The first order density function of $\hat{\mathscr{E}}(X)$ is then given in terms of the variance of τ by

$$f(\mathscr{E}) = \frac{1}{\sqrt{(2\pi)}\sigma_\tau \mathscr{E}} \exp\left\{-\frac{(\ln \mathscr{E})^2}{2\sigma_\tau^2}\right\} U(\mathscr{E}) \tag{10}$$

where $U(\mathscr{E})$ is the unit step function. Equation (10) is the lognormal probability density function [9]. The mean $E\{\hat{\mathscr{E}}\}$, variance $\sigma_\mathscr{E}^2$ and autocovariance $C_\mathscr{E}(X_1-X_2)$ of $\hat{\mathscr{E}}(X)$ are given by

$$E\{\hat{\mathscr{E}}\} = e^{\sigma_\tau^2/2} > 1 \tag{11a}$$

$$\sigma_\mathscr{E}^2 = e^{\sigma_\tau^2}(e^{\sigma_\tau^2}-1) > \sigma_\tau^2 \tag{11b}$$

$$C_\mathscr{E}(X_1-X_2) = e^{\sigma_\tau^2}(e^{C_\tau(X_1-X_2)}-1) \tag{11c}$$

where $C_\tau(X_1-X_2)$ is the autocovariance of $\hat{\tau}(X)$.

Similarly, $\hat{n}(X)$ is assumed to be a function of the random imperfection density $\hat{N}(X)$ only, where $\hat{N}(X)$ is a homogeneous normal random process in X with mean N_0 and variance σ_N^2. Specifically, we have

$$\hat{n}(X) = 1+(n_0-1)\exp\left\{-\frac{n_0'[\hat{N}(X)-N_0]}{n_0-1}\right\} \tag{12a}$$

where

$$n_0 = \hat{n}\Big|_{\hat{N}=N_0} \qquad n_0' = -\frac{d\hat{n}}{d\hat{N}}\Big|_{\hat{N}=N_0} \tag{12b}$$

and thus

$$f(n) = \frac{n_0-1}{\sqrt{(2\pi)}\sigma_N n_0'(n-1)} \exp\left\{-\frac{1}{2}\left(\frac{n_0-1}{\sigma_N n_0'}\right)^2\left[\ln\left(\frac{n-1}{n_0-1}\right)\right]^2\right\} U(n-1). \tag{13}$$

This is a lognormal density function with two parameters (n_0 and $\sigma_N n_0'$) [9], and $\hat{n}(X)$ has the following statistical properties:

$$E\{\hat{n}\} = 1+(n_0-1)\exp\left\{\frac{1}{2}\left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right\} > n_0 \tag{14a}$$

$$\sigma_n^2 = (n_0-1)^2\exp\left\{\left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right\}\left[\exp\left\{\left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right\}-1\right] > (\sigma_N n_0')^2 \tag{14b}$$

$$C_n(X_1-X_2) = (n_0-1)^2\exp\left\{\left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right\}\left[\exp\left\{\frac{n_0'^2}{(n_0-1)^2}C_N(X_1-X_2)\right\}-1\right]. \tag{14c}$$

2.2. *Lateral vibration of axially creeping beam*

As a specific problem consider the free lateral vibration of a nonlinear Maxwell beam, which is subjected initially to a prescribed deterministic axial tensile force followed by a small lateral disturbance at $\bar{T} = 0$. The space coordinate X in this problem will be chosen as the neutral axis of the beam which vibrates in the XZ plane. Thus, $\hat{\sigma}_x(X, Z; \bar{T})$ and $\hat{\varepsilon}_x(X, Z; \bar{T})$ are random functions of X, Z and $\bar{T}$. We make the physically plausible assumption that the deformation of the beam is small. Using the perturbation technique employed in obtaining equation (7), the classical strain–displacement relation and equation of motion for a beam yield the following relations in the first order terms:

$$\tilde{\hat{\varepsilon}}_x(X, Z; \bar{T}) = Z\frac{\partial^2 \tilde{\hat{W}}(X, \bar{T})}{\partial X^2} \tag{15}$$

$$\frac{\partial^2 \tilde{\hat{M}}(X, \bar{T})}{\partial X^2} = -\rho\frac{\partial \tilde{\hat{W}}(X, \bar{T})}{\partial \bar{T}} + Q. \tag{16}$$

In equations (15) and (16) the lateral velocity $\tilde{\hat{W}}(X, \bar{T})$ and bending moment $\tilde{\hat{M}}(X, \bar{T}) = \int_{\bar{A}} \hat{\sigma}_x(X, Z; \bar{T})Z \, \mathrm{d}\bar{A}$ are random functions of X and $\bar{T}$ for a beam with cross-section $\bar{A}$; ρ is the linear density of the beam and Q is an impulsively applied deterministic distributed lateral load which induces a free vibration. Note that, in conformity with the small deformation assumption, the moment due to the axial tensile force P_0 has been neglected.

For convenience, we introduce the following nondimensional quantities:

$$\hat{w}(x, t) = \frac{\tilde{\hat{W}}(X, \bar{T})}{W_m} \qquad \hat{m}(x, t) = \frac{\tilde{\hat{M}}(X, \bar{T})}{W_m \rho L^2}\left(\frac{\sigma_{xi}}{E\dot{\varepsilon}_{c0} n_0}\right)$$

$$q = \frac{Q}{W_m \rho}\left(\frac{\sigma_{xi}}{E\dot{\varepsilon}_{c0} n_0}\right) \qquad t = \bar{T}\left(\frac{E\dot{\varepsilon}_{c0} n_0}{\sigma_{xi}}\right) \qquad x = \frac{X}{L}. \tag{17}$$

In these equations, W_m is the maximum lateral velocity along the beam at the instant $\bar{T} = 0$; L is the length of the beam; and $\hat{w}(x, t)$, $\hat{m}(x, t)$, q, t and x are the nondimensional lateral velocity, bending moment, lateral load, time and axial coordinate respectively.

Two simultaneous differential equations in velocity and bending moment, containing a random process coefficient, may now be obtained from equations (7), (15), (16) and (17) as

$$\frac{\partial \hat{w}(x, t)}{\partial t} = -\frac{\partial^2 \hat{m}(x, t)}{\partial x^2} + q \tag{18}$$

$$\frac{1}{b}\frac{\partial^2 \hat{w}(x, t)}{\partial x^2} = \frac{\partial \hat{m}(x, t)}{\partial t} + \frac{1}{n_0}\hat{\mathscr{E}}(x)\hat{n}(x)\hat{m}(x, t). \tag{19}$$

Here, b is a nondimensional deterministic constant given by

$$b = \frac{\rho L^4}{EI}\left(\frac{E\dot{\varepsilon}_{c0} n_0}{\sigma_{xi}}\right)^2 \tag{20}$$

with I indicating the second moment of area. We observe that $\hat{\mathscr{E}}$ and $\hat{n}$ appear in equation (19) side by side. Furthermore, $\hat{\mathscr{E}}$ and $\hat{n}$ have a similar statistical behavior, i.e., they are both lognormal [equations (10) and (13)]. Thus, the random temperature problem and the random imperfection problem are mathematically equivalent. We shall therefore only present the details for the random temperature problem but will give the final results for both problems.

In Section 3 we shall consider the analysis of equations (18) and (19) for the case where $\hat{\mathscr{E}}$ and $\hat{n}$ are random processes which are only slightly random [case (a)], and then in Section 4 the case where $\hat{\mathscr{E}}$ and $\hat{n}$ are random variables which are largely random [case (b)].

3. CASE (a)—$\hat{\mathscr{E}}$ AND $\hat{\mathbf{n}}$ RANDOM PROCESSES (SLIGHTLY RANDOM)

3.1. *Random temperature problem*

Let us first consider the random temperature problem in which as we have noted only $\hat{\mathscr{E}}(x)$ is random. Letting $\hat{n}(x) = n_0$, equation (19) becomes

$$\frac{1}{b}\frac{\partial^2 \hat{w}(x,t)}{\partial x^2} = \frac{\partial \hat{m}(x,t)}{\partial t} + \hat{\mathscr{E}}(x)\hat{m}(x,t). \tag{21}$$

The simultaneous partial differential equations (18) and (21) can be combined to yield the following two consecutive nonhomogeneous partial differential equations

$$\frac{\partial^2 \hat{m}(x,t)}{\partial t^2} + \hat{\mathscr{E}}(x)\frac{\partial \hat{m}(x,t)}{\partial t} + \frac{1}{b}\frac{\partial^4 \hat{m}(x,t)}{\partial x^4} = \frac{1}{b}\frac{\partial^2 q}{\partial x^2} \tag{22}$$

$$\frac{\partial \hat{w}(x,t)}{\partial t} = -\frac{\partial^2 \hat{m}(x,t)}{\partial x^2} + q. \tag{23}$$

We will assume that the initial conditions at $t = 0^-$ are homogeneous and take the load q as

$$q = \delta(t)\sin k\pi x \qquad k = 1, 2, 3, \ldots \tag{24}$$

where $\delta(t)$ is the Dirac delta function and k is the vibration mode. The physical meaning of these initial conditions and the lateral load will become apparent in the next paragraph.

We may easily convert equations (22) and (23) into two consecutive partial differential equations with zero forcing and nonhomogeneous initial conditions at $t = 0^+$. In so doing we obtain the resulting equations

$$\frac{\partial^2 \hat{m}(x,t)}{\partial t^2} + \hat{\mathscr{E}}(x)\frac{\partial \hat{m}(x,t)}{\partial t} + \frac{1}{b}\frac{\partial^4 \hat{m}(x,t)}{\partial x^4} = 0 \tag{25}$$

$$\frac{\partial \hat{w}(x,t)}{\partial t} = -\frac{\partial^2 \hat{m}(x,t)}{\partial x^2} \tag{26}$$

with the initial conditions

$$\hat{m}(x,0^+) = 0, \qquad \frac{\partial \hat{m}(x,0^+)}{\partial t} = -\frac{(k\pi)^2}{b}\sin k\pi x \qquad k = 1, 2, 3, \ldots \tag{27}$$

and

$$\hat{w}(x,0^+) = \sin k\pi x \qquad k = 1, 2, 3, \ldots, \qquad \frac{\partial \hat{w}(x,0^+)}{\partial t} = 0. \tag{28}$$

Equations (27) and (28) hold with probability one. Thus, we observe that the problem posed in the previous paragraph is equivalent to a free vibration problem with a nonzero deterministic velocity prescribed at $t = 0^+$.

Let us consider a simply-supported beam. Using equations (25) and (26), we obtain the following boundary conditions on $\hat{m}(x,t)$ and $\hat{w}(x,t)$ respectively:

$$\hat{m}(0,t) = \hat{m}(1,t) = \frac{\partial^2 \hat{m}(0,t)}{\partial x^2} = \frac{\partial^2 \hat{m}(1,t)}{\partial x^2} = 0 \tag{29}$$

and

$$\hat{w}(0,t)=\hat{w}(1,t)=\frac{\partial^2\hat{w}(0,t)}{\partial x^2}=\frac{\partial^2\hat{w}(1,t)}{\partial x^2}=0. \tag{30}$$

Again the above relations hold with probability one.

The perturbation method suggested by Keller [8] will now be applied to differential equation (25) with initial and boundary conditions (27) and (29) respectively.

Bending moment. In this section it is assumed that $\hat{\mathscr{E}}(x)$ is homogeneous random process which is slightly random, and we may then write

$$\hat{\mathscr{E}}(x)=1+\varepsilon\hat{\mathscr{E}}_1(x). \tag{31}$$

Here, ε is a small quantity, and $\hat{\mathscr{E}}_1(x)$ is a homogeneous random process. Substituting equation (31) into equation (25), we obtain

$$(L_0+\varepsilon\hat{L}_1)\hat{m}(x,t)=0 \tag{32}$$

where

$$L_0=\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{1}{b}\frac{\partial^4}{\partial x^4} \tag{33}$$

$$\hat{L}_1=\hat{\mathscr{E}}_1(x)\frac{\partial}{\partial t}. \tag{34}$$

The solution to equation (32) is not separable in x and t, and is extremely difficult to obtain. However, utilizing the perturbation method given in [8], we may obtain an approximate expression for the mean bending moment as

$$E\{\hat{m}\}=(1-\varepsilon L_0^{-1}E\{\hat{L}_1\}+\varepsilon^2L_0^{-1}E\{\hat{L}_1L_0^{-1}\hat{L}_1\})m_0. \tag{35}$$

Here, m_0 is the solution to the unperturbed deterministic linear partial differential equation

$$\frac{\partial^2 m_0}{\partial t^2}+\frac{\partial m_0}{\partial t}+\frac{1}{b}\frac{\partial^4 m_0}{\partial x^4}=0 \tag{36}$$

with the prescribed deterministic initial and boundary conditions [equations (27) and (29)]. Also, in equation (35) L_0^{-1} is the inverse operator associated with L_0, and is thus an integral operator whose kernel is the Green's function $G(x,\xi;t)$ satisfying

$$\frac{\partial^2 G}{\partial t^2}+\frac{\partial G}{\partial t}+\frac{1}{b}\frac{\partial^4 G}{\partial x^4}=\delta(t)\delta(x-\xi) \tag{37}$$

with homogeneous initial conditions at $t=0^-$. Note that in equation (35) those terms of order ε and ε^2 satisfy homogeneous initial and boundary conditions.

The solution to equation (36) can be assumed in the form

$$m_0=T_1(t)\sin k\pi x \qquad k=1,2,3,\ldots \tag{38}$$

where $T_1(t)$ is a function of time t only. Note that the boundary conditions (29) are automatically satisfied by equation (38). Substituting equation (38) into equation (36) yields an ordinary differential equation in T_1. We now make the additional assumption that the vibration is underdamped. Thus, setting $(k\pi)^4/b - \frac{1}{4} > 0$ and using equations (27), we obtain

$$m_0 = -\frac{(k\pi)^2}{br_k} e^{-t/2} \sin r_k t \sin k\pi x \qquad k = 1, 2, 3, \ldots \tag{39a}$$

where the nondimensional frequency is given by

$$r_k = \sqrt{\left(\frac{(k\pi)^4}{b} - \frac{1}{4}\right)}. \tag{39b}$$

Next, we must obtain the Green's function satisfying equation (37). A comparison of equations (36) and (37) indicates that the Green's function is equal to the deterministic bending moment which results from a unit impulse applied to the beam at the location $x = \xi$ at time $t = 0$. This Green's function vanishes for either $t < 0$ or $x, \xi \notin [0, 1]$. The normal modes obtained from the homogeneous part of equation (37) with the prescribed homogeneous boundary conditions are given by the orthogonal functions

$$h_j(x) = \sin j\pi x. \tag{40}$$

We now try a solution to equation (37) in the form

$$G(x, \xi; t) = \sum_{j=1}^{\infty} \eta_j(\xi, t) h_j(x) \tag{41}$$

whereby

$$\sum_{j=1}^{\infty} \frac{d^2\eta_j}{dt^2} h_j + \sum_{j=1}^{\infty} \frac{d\eta_j}{dt} h_j + \frac{1}{b} \sum_{j=1}^{\infty} (j\pi)^4 \eta_j h_j = \delta(t)\delta(x-\xi). \tag{42}$$

The orthogonality of the normal modes $h_j(x)$ can be used to reduce equation (42) to the system of uncoupled equations

$$\frac{d^2\eta_j}{dt^2} + \frac{d\eta_j}{dt} + \frac{(j\pi)^4}{b}\eta_j = 2\delta(t) \sin j\pi\xi \qquad j = 1, 2, 3, \ldots. \tag{43}$$

Solving equation (43) for η_j, we obtain with the use of equation (41)

$$G(x, \xi; t) = \begin{cases} \displaystyle\sum_{j=1}^{\infty} \frac{2}{r_j} e^{-t/2} \sin r_j t \sin j\pi\xi \sin j\pi x, & t \geq 0, 0 \leq x, \xi \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{44}$$

where r_j is defined by equation (39b). Finally, with this Green's function the inverse operator L_0^{-1} operating on some function $F(x, t)$ can be given by

$$L_0^{-1} F(x, t) = \int_0^t \int_0^1 G(x, \xi; t - \tau) F(\xi, \tau) \, d\xi \, d\tau. \tag{45}$$

Let us now return to equation (35). In order to evaluate the terms of order ε and ε^2, we must first determine the mean $\bar{\mathscr{E}}_1$ and mean square $\overline{\mathscr{E}_1^2}$ of $\hat{\mathscr{E}}_1(x)$ for $\hat{\tau}(x)$ a homogeneous random process with mean zero and variance σ_τ^2. These follow directly from equations (11) and (31) as

$$\bar{\mathscr{E}}_1 = \frac{e^{\sigma_\tau^2/2}-1}{\varepsilon} \qquad \overline{\mathscr{E}_1^2} = \frac{e^{2\sigma_\tau^2}-2\,e^{\sigma_\tau^2/2}+1}{\varepsilon^2}. \tag{46}$$

For the term of order ε^2 we must also specify the autocovariance of $\hat{\tau}(x)[C_\tau(\eta)$ where $\eta = x_1 - x_2]$. The analysis would become extremely complicated if we were to choose an exponential expression for $C_\tau(\eta)$ as in [5] and [6]. On the other hand, the analysis is much simpler if we choose an exponential function for the autocorrelation of $\hat{\mathscr{E}}_1(x)$, i.e.

$$E\{\hat{\mathscr{E}}_1(x_1)\hat{\mathscr{E}}_1(x_2)\} = \overline{\mathscr{E}_1^2}\,e^{-|\eta|/d} \tag{47}$$

where d is the nondimensional correlation distance. The autocovariance of $\hat{\tau}(x)$ is then obtained in inverse fashion from equations (11c), (31), (46) and (47) as

$$C_\tau(\eta) = \ln[2\,e^{\sigma_\tau^2/2}-1+(e^{2\sigma_\tau^2}-2\,e^{\sigma_\tau^2/2}+1)\,e^{-|\eta|/d}]-\sigma_\tau^2. \tag{48}$$

The following limiting values follow from this result:

$$\lim_{\eta\to 0} C_\tau(\eta) = \sigma_\tau^2 \tag{49}$$

$$\lim_{\eta\to\infty} C_\tau(\eta) = \ln(2\,e^{\sigma_\tau^2/2}-1)-\sigma_\tau^2. \tag{50}$$

Equation (50) gives a very small negative value for $\eta \to \infty$, since $\hat{\tau}$ has been assumed to be slightly random (i.e., σ_τ is small). Equation (48) has been plotted in Fig. 1 for typical values $\sigma_\tau = 0{\cdot}1$ and $d = 0{\cdot}2$, and we note that we still get a physically plausible exponential-like function for C_τ.

Using the deterministic bending moment m_0 [equations (39)], the Green's function $G(x, \xi; t)$ [equation (44)], and equations (45)–(47), we may then proceed to evaluate equation (35) and thereby obtain the rather complicated result

$$\begin{aligned}
E\{\hat{m}\} = {} & -\frac{(k\pi)^2}{br_k}\,e^{-t/2}\sin r_k t\sin k\pi x+\varepsilon\frac{(k\pi)^2\bar{\mathscr{E}}_1}{2br_k^2}\,e^{-t/2}\left[\left(-\frac{1}{2r_k}+r_k t\right)\sin r_k t\right.\\
& \left.+t/2\cos r_k t\right]\sin k\pi x-\varepsilon^2\frac{2(k\pi)^2}{br_k}\overline{\mathscr{E}_1^2}\,e^{-t/2}\Big\{d_1[(A_1+A_2t+A_3t^2)\sin r_k t\\
& +(A_4t+A_5t^2)\cos r_k t]\sin k\pi x+\sum_{\substack{j=1\\ j\neq k}}^{\infty}\left[d_2^{(j)}\sin k\pi x+\left(d_3^{(j)}-\frac{d_4}{4}\delta_{j(3k)}\right)\sin j\pi x\right]\\
& \cdot[(A_6^{(j)}+A_7^{(j)}t)\sin r_k t+(A_8^{(j)}+A_9^{(j)}t)\cos r_k t+A_{10}^{(j)}\sin r_j t+A_{11}^{(j)}\cos r_j t]\\
& +\sum_{\substack{j=1\\ j\neq k}}^{\infty}(2d_5^{(j)}-d_6^{(j)}\delta_{(3j)k})[A_{12}^{(j)}\sin r_k t+(A_{13}^{(j)}+A_{14}^{(j)}t)\sin r_j t+(A_{15}^{(j)}+A_{16}^{(j)}t)\cos r_j t\\
& +A_{17}^{(j)}\cos r_k t]\sin j\pi x+\sum_{\substack{i=1\\ i\neq k}}^{\infty}\sum_{\substack{j=1\\ \neq j\neq i}}^{\infty}(2d_7^{(ij)}-d_6^{(j)}\delta_{i(k-2j)}-d_8^{(j)}\delta_{i(k+2j)})(A_{18}^{(ij)}\sin r_k t\\
& +A_{19}^{(ij)}\sin r_i t+A_{20}^{(ij)}\sin r_j t+A_{21}^{(ij)}\cos r_k t+A_{22}^{(ij)}\cos r_i t+A_{23}^{(ij)}\cos r_j t)\sin i\pi x\Big\}
\end{aligned} \tag{51}$$

where δ_{ij} is the Kronecker delta.

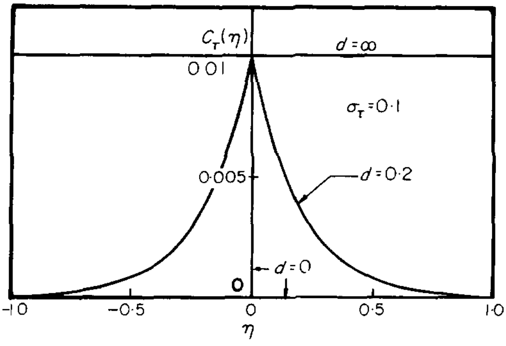

FIG. 1. Autocovariance of $\hat{t}$—case (a).

In equation (51)

$$A_1 = \frac{4r_k^2+3}{16r_k^4} \qquad A_2 = -\frac{1}{4r_k^2}$$

$$A_3 = \frac{4r_k^2-1}{16r_k^2} \qquad A_4 = -r_k A_1$$

$$A_5 = -r_k A_2 \qquad A_6^{(j)} = \frac{(4r_k^2-1)(r_k^2-r_j^2)-2r_k^2(1-4r_j^2)}{4r_k^2(r_k^2-r_j^2)^2}$$

$$A_7^{(j)} = \frac{1}{r_k^2-r_j^2} \qquad A_8^{(j)} = \frac{2r_k}{(r_k^2-r_j^2)^2}$$

$$A_9^{(j)} = -\frac{4r_k^2-1}{4r_k(r_k^2-r_j^2)} \qquad A_{10}^{(j)} = \frac{r_k(1-4r_j^2)}{2r_j(r_k^2-r_j^2)^2}$$

$$A_{11}^{(j)} = -A_8^{(j)} \qquad A_{12}^{(j)} = -\frac{4r_k^2-1}{4(r_k^2-r_j^2)^2}$$

$$A_{13}^{(j)} = \frac{r_k[2r_j^2(4r_k^2-1)+(1-4r_j^2)(r_k^2-r_j^2)]}{8r_j^3(r_k^2-r_j^2)^2} \qquad A_{14}^{(j)} = \frac{-r_k}{2r_j(r_k^2-r_j^2)}$$

$$A_{15}^{(j)} = \frac{A_8^{(j)}}{2} \qquad A_{16}^{(j)} = \frac{-r_k(1-4r_j^2)}{8r_j^2(r_k^2-r_j^2)}$$

$$A_{17}^{(j)} = -A_{15}^{(j)} \qquad A_{18}^{(ij)} = -\frac{4r_k^2-1}{4(r_k^2-r_j^2)(r_k^2-r_i^2)}$$

$$A_{19}^{(ij)} = \frac{r_k(1-4r_i^2)}{4r_i(r_k^2-r_i^2)(r_j^2-r_i^2)} \qquad A_{20}^{(ij)} = -\frac{r_k(1-4r_j^2)}{4r_j(r_j^2-r_i^2)(r_k^2-r_j^2)}$$

$$A_{21}^{(ij)} = -\frac{r_k}{(r_k^2-r_j^2)(r_k^2-r_i^2)} \qquad A_{22}^{(ij)} = -\frac{r_k}{(r_k^2-r_i^2)(r_j^2-r_i^2)}$$

$$A_{23}^{(ij)} = \frac{r_k}{(r_k^2-r_j^2)(r_j^2-r_i^2)} \tag{52}$$

and

$$
\begin{aligned}
d_1 &= \frac{3+8d^2k^2\pi^2}{4dC_1}+\frac{8d^2k^4\pi^4}{C_1^2}(\mathrm{e}^{-1/d}-1)\\
d_2^{(j)} &= \frac{1/d^2+(k^2+j^2)\pi^2}{2dC_2^{(j)}C_3^{(j)}}+\frac{8k^2j^2\pi^4[(-1)^{k-j}\,\mathrm{e}^{-1/d}-1]}{(dC_2^{(j)}C_3^{(j)})^2}\\
d_3^{(j)} &= \frac{4k^3j\pi^4[1+(-1)^{k-j}](\mathrm{e}^{-1/d}-1)}{C_1C_2^{(j)}C_3^{(j)}}\\
d_4 &= \frac{1}{dC_1}\\
d_5^{(j)} &= \frac{4kj^3\pi^4[1+(-1)^{k-j}](\mathrm{e}^{-1/d}-1)}{C_2^{(j)}C_3^{(j)}(1/d^2+4j^2\pi^2)}\\
d_6^{(j)} &= \frac{1}{2dC_3^{(j)}}\\
d_7^{(ij)} &= \frac{4ij^2k\pi^4[((-1)^{i-j}+(-1)^{k-j})\,\mathrm{e}^{-1/d}-(1+(-1)^{i+k-2j})]}{d^2C_2^{(j)}C_3^{(j)}(1/d^2+(i-j)^2\pi^2)(1/d^2+(i+j)^2\pi^2)}\\
d_8^{(j)} &= \frac{1}{2dC_2^{(j)}}
\end{aligned}
\tag{53}
$$

where

$$
C_1 = \frac{1}{d^2}+4k^2\pi^2
$$

$$
\begin{matrix} C_2^{(j)} \\ C_3^{(j)} \end{matrix} = \frac{1}{d^2}+(k\pm j)^2\pi^2. \tag{54}
$$

As a particular example we will consider the beam treated deterministically in [7], namely a uniform rectangular cross-section beam made of aluminum alloy 1100. The cross-sectional dimensions are taken to be 2 in in width and 3 in in depth, and the length of the beam is taken as 36 in. The mean temperature of the beam is chosen as 260°C and the axial stress $\sigma_{xi} = \sigma_c$ is taken equal to 3000 psi. The material properties are given in Table 1 (see [7]).

TABLE 1. MATERIAL PROPERTIES OF ALUMINUM ALLOY 1100 (260°C)

$\rho = 1{\cdot}2 \times 10^{-10}$	lb-h^2-in^{-2}
$E = 8{\cdot}0 \times 10^6$	psi
$n_0 = 4{\cdot}55$	
$\lambda_0 = 3{\cdot}64 \times 10^3$	psi-h$^{1/n_0}$

Using the data given in the previous paragraph and in Table 1 and taking the vibration mode k to be one, we obtain the numerical values given in Table 2.

TABLE 2. NUMERICAL VALUES

$I = 4{\cdot}5$	in^4	
$\dot{\varepsilon}_{c0} = \left(\frac{\sigma_{xi}}{\lambda_0}\right)^{n_0} = 0{\cdot}415$	hr^{-1}	[equation (4)]
$b = \frac{\rho L^4}{EI}\left(\frac{E\dot{\varepsilon}_{c0}n_0}{\sigma_{xi}}\right)^2 = 1{\cdot}419\times 10^{-4}$		[equation (20)]
$r_k = \sqrt{\left(\frac{(k\pi)^4}{b}-\frac{1}{4}\right)} = 828{\cdot}68$ for $k = 1$		[equation (39b)]

It may be seen from Table 2 that, as a consequence of the small damping in this example ($b \ll 1$), $r_1 \gg 1$ and $|r_j^2 - r_i^2| \gg 1$ for $j \neq i$. We then find that, for any values of σ_τ and d, the expression of order ε in equation (51) is completely dominated by the $r_k t \sin r_k t$ term and the expression of order ε^2 is completely dominated by the $d_1 A_3 t^2 \sin r_k t$ term. Thus, we may now considerably simplify the expression for $E\{\hat{m}\}$, with negligible loss in accuracy, to the following form which is separated in x and t:

$$E\{\hat{m}\} = -\frac{\pi^2}{br_1}\left(1 - \varepsilon\frac{\bar{\mathscr{E}}_1 t}{2} + \varepsilon^2 2d_1 A_3 \overline{\mathscr{E}_1^2} t^2\right) \mathrm{e}^{-t/2} \sin r_1 t \sin \pi x. \tag{55}$$

With the use of equations (17), (20), (39b) and (52), equation (55) may be rewritten in terms of the physical parameters and dimensional time $\bar{T}$ (in seconds) as

$$E\left\{\frac{\hat{m}}{m^*}\right\} = (1 - \varepsilon\bar{\mathscr{E}}_1 \alpha \bar{T} + \varepsilon^2 2d_1 \overline{\mathscr{E}_1^2} \alpha^2 \bar{T}^2)\, \mathrm{e}^{-\alpha \bar{T}} \sin \beta \bar{T} \sin \pi x \tag{56}$$

where

$$\alpha = \frac{1}{7200}\left(\frac{E\dot{\varepsilon}_{c0} n_0}{\sigma_{xi}}\right) \qquad \beta = \frac{1}{3600}\left(\frac{\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho}} \tag{57a}$$

and where m^* is a normalizing factor defined as

$$m^* = -\frac{\beta}{2\pi^2 \alpha}. \tag{57b}$$

In the present example $\alpha = 0{\cdot}7$ and $\beta = 1158{\cdot}7$, and thus we see that as a consequence of small damping $\beta \gg \alpha$.

For practical purposes it is sufficient to confine our attention to the time interval over which the exponential function in equation (56) decays to one per cent of its initial value, i.e., $0 \leq \bar{T} \leq 4{\cdot}6/\alpha$. Within this interval, we will consider that the "slightly random" assumption implies that $\varepsilon^2 2d_1 \overline{\mathscr{E}_1^2} \alpha^2 \bar{T}^2 \ll 1$ in equation (56). This requires that

$$\varepsilon^2 \overline{\mathscr{E}_1^2} = \mathrm{e}^{2\sigma_\tau^2} - 2\,\mathrm{e}^{\sigma_\tau^2/2} + 1 \ll \left\{42{\cdot}32\left[\frac{3 + 8d^2\pi^2}{4d(1/d^2 + 4\pi^2)} + \frac{8d^2\pi^4(\mathrm{e}^{-1/d} - 1)}{(1/d^2 + 4\pi^2)^2}\right]\right\}^{-1} \tag{58}$$

Equation (58) will be used as a condition on σ_τ for a given d.

In Fig. 2(a), a single curve showing the decay of $E\{\hat{m}/m^*)_{\text{LMP}}$ (LMP = locus of maximum points) for the mean bending moment [equation (56)]] at the midspan of the beam ($x = \frac{1}{2}$) is plotted vs time $\bar{T}$ in seconds for typical values $\sigma_\tau = 0{\cdot}1$ and $d = 0, 0{\cdot}2, \infty$—all of which

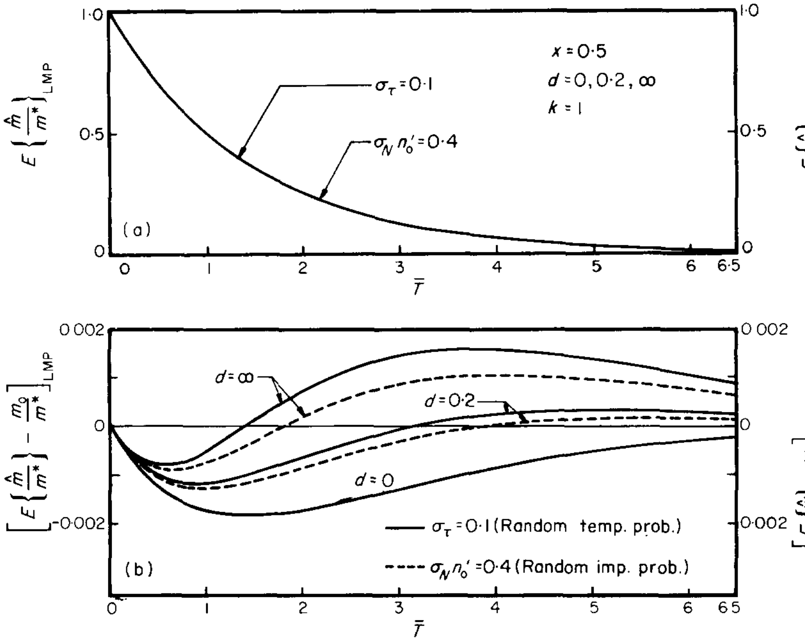

FIG. 2. Decay curves—case (a). (a) LMP of mean bending moment and velocity. (b) deviation from LMP of deterministic bending moment and velocity.

satisfy condition (58). The deviation of the LMP of the mean bending moment from the LMP of the deterministic bending moment, $[E\{\hat{m}/m^*\} - m_0/m^*]_{LMP}$, has been plotted in Fig. 2(b) (solid curves). Note that for $d > 0$ this deviation vanishes at a root $\bar{T} = \bar{T}_r$, and that the LMP of the mean bending moment is less than the LMP of the deterministic bending moment in the interval $0 < \bar{T} < \bar{T}_r$, and is greater for $\bar{T} > \bar{T}_r$. We also see that there exist two critical times $\bar{T}_{min}$ and $\bar{T}_{max}$ where the random effect is an extremum. We obtain $\bar{T}_r$ and these two critical times from equations (56) and (57) as

$$\bar{T}_r = \frac{\bar{\mathscr{E}}_1}{2\varepsilon d_1 \bar{\mathscr{E}}_1^2 \alpha} \tag{59a}$$

$$\begin{matrix}\bar{T}_{min} \\ \bar{T}_{max}\end{matrix} = \frac{\bar{T}_r}{2} + \frac{1}{\alpha} \mp \sqrt{\left(\frac{\bar{T}_r^2}{4} + \frac{1}{\alpha^2}\right)}. \tag{59b}$$

With $\sigma_\tau = 0{\cdot}1$ and $d = 0{\cdot}2$ in this example equations (59) yield $\bar{T}_r = 3{\cdot}147$ sec, $\bar{T}_{min} = 0{\cdot}877$ sec and $\bar{T}_{max} = 5{\cdot}131$ sec, which approximately equal 580, 161 and 946 cycles of vibration respectively. Also, for these values equations (11a) yield $E\{\hat{\mathscr{E}}\} = 1{\cdot}005$, and Fig. 2 yields $E\{\hat{m}/m_0\}_{LMP} = 0{\cdot}9977$ at $\bar{T} = \bar{T}_{min} = 0{\cdot}877$ sec. Thus, the bending moment is sensitive to small variations in temperature.

Velocity. Turning now to the velocity, we obtain from equation (26)

$$\frac{\partial}{\partial t}(E\{\hat{w}\}) = -\frac{\partial^2}{\partial x^2}(E\{\hat{m}\}) \tag{60}$$

for $\hat{w}(x, t)$ and $\hat{m}(x, t)$ mean square differentiable with respect to time t and axial coordinate x respectively. Substituting equation (51) into equation (60), integrating the result with respect to t, and then using initial conditions (28), we obtain

$$
\begin{aligned}
E\{\hat{w}\} = {} & \frac{1}{r_k} e^{-t/2}(\tfrac{1}{2}\sin r_k t + r_k \cos r_k t)\sin k\pi x + \varepsilon \frac{(k\pi)^4 \bar{\mathscr{E}}_1}{2br_k^2} e^{-t/2}\left(\frac{1}{r_k}\sin r_k t - t\cos r_k t\right)\sin k\pi x \\
& - \varepsilon^2 \frac{2(k\pi)^2}{br_k}\overline{\mathscr{E}_1^2}\, e^{-t/2}\Bigg\{ d_1 (k\pi)^2[(B_1 + B_2 t + B_3 t^2)\sin r_k t + (B_4 t + B_5 t^2)\cos r_k t]\sin k\pi x \\
& + \sum_{\substack{j=1\\ j\neq k}}^{\infty}\left[d_2^{(j)}(k\pi)^2 \sin k\pi x + \left(d_3^{(j)} - \frac{d_4}{4}\delta_{j(3k)}\right)(j\pi)^2 \sin j\pi x\right][(B_6^{(j)} + B_7^{(j)} t)\sin r_k t \\
& + (B_8^{(j)} + B_9^{(j)} t)\cos r_k t + B_{10}^{(j)} \sin r_j t + B_{11}^{(j)}\cos r_j t] + \sum_{\substack{j=1\\ j\neq k}}^{\infty} (2d_5^{(j)} - d_6^{(j)}\delta_{(3j)k} \\
& \cdot (j\pi)^2 [B_{12}^{(j)} \sin r_k t + (B_{13}^{(j)} + B_{14}^{(j)} t)\sin r_j t + (B_{15}^{(j)} + B_{16}^{(j)} t)\cos r_j t + B_{17}^{(j)}\cos r_k t] \\
& \cdot \sin j\pi x + \sum_{\substack{i=1\\ i\neq k\neq j\neq i}}^{\infty}\sum_{j=1}^{\infty} (2d_7^{(ij)} - d_6^{(j)}\delta_{i(k-2j)} - d_8^{(j)}\delta_{i(k+2j)})(i\pi)^2 (B_{18}^{(ij)}\sin r_k t \\
& + B_{19}^{(ij)}\sin r_i t + B_{20}^{(ij)}\sin r_j t + B_{21}^{(ij)}\cos r_k t + B_{22}^{(ij)}\cos r_i t + B_{23}^{(ij)}\cos r_j t)\sin i\pi x \Bigg\}.
\end{aligned}
\tag{61}
$$

In equation (61)

$$
\begin{aligned}
B_1 &= \frac{-3}{8r_k^4} & B_2 &= \frac{1}{4r_k^2} \\
B_3 &= \frac{B_2}{2} & B_4 &= -r_k B_1 \\
B_5 &= -r_k B_2 & B_6^{(j)} &= \frac{(3r_k^2 - r_j^2)}{2r_k^2(r_k^2 - r_j^2)^2} \\
B_7^{(j)} &= \frac{-1}{r_k^2 - r_j^2} & B_8^{(j)} &= \frac{-2r_k}{(r_k^2 - r_j^2)^2} \\
B_9^{(j)} &= \frac{B_7^{(j)}}{2r_k} & B_{10}^{(j)} &= \frac{B_8^{(j)}}{2r_j} \\
B_{11}^{(j)} &= -B_8^{(j)} & B_{12}^{(j)} &= \frac{-1}{2(r_k^2 - r_j^2)^2} \\
B_{13}^{(j)} &= -\frac{r_k(r_k^2 - 3r_j^2)}{4r_j^3(r_k^2 - r_j^2)^2} & B_{14}^{(j)} &= \frac{r_k}{2r_j(r_k^2 - r_j^2)} \\
B_{15}^{(j)} &= 2r_k B_{12}^{(j)} & B_{16}^{(j)} &= \frac{B_{14}^{(j)}}{2r_j} \\
B_{17}^{(j)} &= -B_{15}^{(j)} & B_{18}^{(ij)} &= -\frac{1}{2(r_k^2 - r_j^2)(r_k^2 - r_i^2)}
\end{aligned}
$$

$$B_{19}^{(ij)} = -\frac{r_k}{2r_i(r_k^2-r_i^2)(r_j^2-r_i^2)} \qquad B_{20}^{(ij)} = \frac{r_k}{2r_j(r_j^2-r_i^2)(r_k^2-r_j^2)}$$

$$B_{21}^{(ij)} = \frac{r_k}{(r_k^2-r_j^2)(r_k^2-r_i^2)} \qquad B_{22}^{(ij)} = \frac{r_k}{(r_k^2-r_i^2)(r_j^2-r_i^2)}$$

$$B_{23}^{(ij)} = -\frac{r_k}{(r_j^2-r_i^2)(r_k^2-r_j^2)}. \tag{62}$$

Again as a specific example we will consider the beam with small damping discussed in [7]. Proceeding as in our analysis of the bending moment, we find that the expression of order 1 in equation (61) is completely dominated by the $r_k \cos r_k t$ term, the expression of order ε is completely dominated by the $t \cos r_k t$ term, and the expression of order ε^2 is completely dominated by the $d_1 B_5 t^2 \cos r_k t$ term. Thus, we may in this case considerably simplify equation (61) for $E\{\hat{w}\}$, with negligible loss in accuracy, to the following form:

$$E\{\hat{w}\} = (1-\varepsilon\bar{\mathscr{E}}_1\alpha\bar{T}+\varepsilon^2 2d_1\overline{\mathscr{E}_1^2}\alpha^2\bar{T}^2)\,e^{-\alpha\bar{T}}\cos\beta\bar{T}\sin\pi x. \tag{63}$$

Here, $\bar{T}$ is the time in seconds, and α and β are as defined in equations (57).

The expression for $E\{\hat{w}\}$ given by equation (63) is identical with the expression for $E\{\hat{m}/m^*\}$ given by equation (56) except that $\sin\beta\bar{T}$ is replaced by $\cos\beta\bar{T}$. Since $\beta \gg 1$, these two expressions give essentially the same locus of the maximum points. Thus, Fig. 2 also applies in this case. The velocity is clearly sensitive to small fluctuations in temperature.

3.2. *Random imperfection problem*

As we have previously stated, the random imperfection problem and the random temperature problem are mathematically equivalent. Instead of using equation (31) we start with the relation

$$\hat{n}(x) = n_0[1+\varepsilon\hat{n}_1(x)] \tag{64}$$

where $\hat{n}_1(x)$ is a homogeneous random process. Its mean $\bar{n}_1$ and mean square $\overline{n_1^2}$ are obtained from equations (14) and (64) as

$$\bar{n}_1 = \frac{(n_0-1)\left[\exp\left\{\frac{1}{2}\left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right\}-1\right]}{n_0\varepsilon} \tag{65}$$

$$\overline{n_1^2} = \frac{(n_0-1)^2\left[\exp\left\{2\left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right\}-2\exp\left\{\frac{1}{2}\left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right\}+1\right]}{n_0^2\varepsilon^2}. \tag{66}$$

The autocorrelation of $\hat{n}_1(x)$ is assumed to be

$$E\{\hat{n}_1(x_1)\hat{n}_1(x_2)\} = \overline{n_1^2}\,e^{-|\eta|/d} \tag{67}$$

based upon the same argument given in Section 3.1.

Proceeding as in the random temperature problem, we obtain the same results [i.e., equations (51), (56), (58), (59), (61) and (63)] with the substitutions

$$\bar{\mathscr{E}}_1 \to \bar{n}_1 \qquad \overline{\mathscr{E}_1^2} \to \overline{n_1^2}. \tag{68}$$

The curve shown in Fig. 2(a) also applies in this problem for values $\sigma_N n_0' = 0{\cdot}4$ and $d = 0$, $0{\cdot}2$, ∞—all of which satisfy condition (58) with the substitutions (68). The deviation of the LMP of the mean bending moment and of the mean velocity from the LMP of the deterministic bending moment and velocity have also been plotted in Fig. 2(b) (dashed curves). For the values $\sigma_N n_0' = 0{\cdot}4$ and $d = 0{\cdot}2$, equations (59) with the substitutions (68) yields $E\{\hat{n}/n_0\} = 1{\cdot}005$, and Fig. 2 yields $E\{\hat{m}/m_0\}_{\text{LMP}} = E\{\hat{w}/w_0\}_{\text{LMP}} = 0{\cdot}9974$ at $T = T_{\min}$ 740, 179 and 1088 cycles of vibration respectively. Also, for these values equation (14a) yields $E\{\hat{n}/n_0\} = 1{\cdot}005$, and Fig. 2 yields $E\{\hat{m}/m_0\}_{\text{LMP}} = E\{\hat{w}/w_0\}_{\text{LMP}} = 0{\cdot}9974$ at $T = T_{\min}$ $= 0{\cdot}973$ sec. Thus, the bending moment and the velocity are also sensitive to small variations in imperfection density.

We conclude that both bending moment and velocity are sensitive to small fluctuations in temperature and in imperfection density. This differs from the results obtained for the 3-bar truss in [5] and statically loaded beam in [6], where the stress was essentially deterministic and only the velocity was sensitive to fluctuations in temperature and in imperfection density.

4. CASE (b)—$\hat{\mathscr{E}}$ AND $\hat{\mathbf{n}}$ RANDOM VARIABLES (LARGELY RANDOM)

4.1. *Random temperature problem*

Again let us first consider the random temperature problem in which only $\hat{\mathscr{E}}$ is random. In this section $\hat{\mathscr{E}}$ is taken to be a random variable which is largely random. Thus, whereas in equation (31) ε may no longer be used as a perturbation parameter, $\hat{\mathscr{E}}_1(x)$ is simply a random variable $\hat{\mathscr{E}}_1$. It follows that the consecutive differential equations (25) and (26) may be simplified to two uncoupled linear differential equations in $\hat{m}(x, t)$ and $\hat{w}(x, t)$, each with a random constant coefficient, i.e.,

$$\frac{\partial^2 \hat{m}(x, t)}{\partial t^2} + \hat{\mathscr{E}} \frac{\partial \hat{m}(x, t)}{\partial t} + \frac{1}{b} \frac{\partial^4 \hat{m}(x, t)}{\partial x^4} = 0 \tag{69}$$

$$\frac{\partial^2 \hat{w}(x, t)}{\partial t^2} + \hat{\mathscr{E}} \frac{\partial \hat{w}(x, t)}{\partial t} + \frac{1}{b} \frac{\partial^4 \hat{w}(x, t)}{\partial x^4} = 0. \tag{70}$$

The deterministic initial conditions are given by equations (27) and (28), and the deterministic homogeneous boundary conditions are again given by equations (29) and (30). Note that whereas equations (69) and (70) are identical, the initial conditions on $\hat{m}$ and $\hat{w}$ are different. We shall only present the details of the velocity solution but will give the final results for both velocity and bending moment.

Velocity. The solution to equation (70) is separable, and may be expressed as the product of a random function of time and a deterministic function of space in the form

$$\hat{w}(x, t) = \hat{T}_2(t) \sin k\pi x \qquad k = 1, 2, 3, \ldots, \tag{71}$$

Substituting equation (71) into equation (70), solving the resulting ordinary differential equation in $\hat{T}_2(t)$ with the initial conditions (28), and using equations (17), (20) and (57), we obtain the velocity for the underdamped case ($\hat{\mathscr{E}} < \beta k^2/\alpha$) as

$$\hat{w}(x, t) = \frac{\beta k^2}{\alpha\sqrt{[(\beta k^2/\alpha)^2 - \hat{\mathscr{E}}^2]}} e^{-\alpha\hat{\mathscr{E}}\bar{T}} \sin(\alpha\sqrt{[(\beta k^2/\alpha)^2 - \hat{\mathscr{E}}^2]}\bar{T} + \hat{\theta}) \sin k\pi x \quad k = 1, 2, 3, \ldots \tag{72}$$

where the phase angle is given by

$$\hat{\theta} = \tan^{-1}\left(\frac{\sqrt{[(\beta k^2/\alpha)^2-\hat{\mathscr{E}}^2]}}{\hat{\mathscr{E}}}\right). \tag{73}$$

Since there is zero probability that $\hat{\mathscr{E}}$ in this case is greater than $\beta k^2/\alpha$, the lognormal density function for $\hat{\mathscr{E}}$ [equation (10)] must be replaced by a truncated lognormal density function. Thus, we let

$$f(\mathscr{E}) = \frac{1}{\Omega_1\sqrt{(2\pi)}\sigma_\tau\mathscr{E}}\exp\left\{-\frac{(\ln\mathscr{E})^2}{2\sigma_\tau^2}\right\}[U(\mathscr{E})-U(\mathscr{E}-\beta k^2/\alpha)] \tag{74a}$$

and

$$\Omega_1 = \tfrac{1}{2}+\operatorname{erf}\left[\frac{\ln(\beta k^2/\alpha)}{\sigma_\tau}\right] \tag{74b}$$

where Ω_1 is such that equation (74a) satisfies the normalization condition. Note that $\operatorname{erf} x$ is the error function defined as

$$\operatorname{erf} x = \frac{1}{\sqrt{(2\pi)}}\int_0^x e^{-y^2/2}\,dy. \tag{75}$$

The mean $E\{\hat{\mathscr{E}}\}$ and variance $\sigma_{\mathscr{E}}^2$ based on the truncated lognormal density function [equation (74a)] are given by

$$E\{\hat{\mathscr{E}}\} = \frac{H_1}{\Omega_1}e^{\sigma_\tau^2/2} \tag{76a}$$

and

$$\sigma_{\mathscr{E}}^2 = \left(\frac{H_1}{\Omega_1}\right)^2 e^{\sigma_\tau^2}\left(\frac{\Omega_1 H_2}{H_1^2}e^{\sigma_\tau^2}-1\right) \tag{76b}$$

where

$$H_1 = \tfrac{1}{2}+\operatorname{erf}\left(\frac{\ln(\beta k^2/\alpha)-\sigma_\tau^2}{\sigma_\tau}\right) \tag{77a}$$

$$H_2 = \tfrac{1}{2}+\operatorname{erf}\left(\frac{\ln(\beta k^2/\alpha)-2\sigma_\tau^2}{\sigma_\tau}\right). \tag{77b}$$

Expressions for the first order density function and the statistical moments of $\hat{w}(x,t)$ may be found from equations (72) and (74). For example, the mean velocity is given by

$$E\{\hat{w}\} = \frac{\beta k^2 \sin k\pi x}{\alpha\Omega_1\sqrt{(2\pi)}\sigma_\tau}\int_0^{\beta k^2/\alpha}\frac{\exp\left\{-\left(\alpha\mathscr{E}\bar{T}+\frac{(\ln\mathscr{E})^2}{2\sigma_\tau^2}\right)\right\}}{\mathscr{E}\sqrt{[(\beta k^2/\alpha)^2-\mathscr{E}^2]}}\sin(\alpha\sqrt{[(\beta k^2/\alpha)^2-\mathscr{E}^2]}\bar{T}+\theta)\,d\mathscr{E}. \tag{78}$$

Decay curves for the mean velocity [equation (78)] at the midspan of the beam discussed in the previous section with $k = 1$, and $\sigma_\tau = 0{\cdot}2$ and $0{\cdot}5$ are presented in Fig. 3. In this example $\beta \gg \alpha$ (small damping), and as a result the normalization factor Ω_1 is very close to unity [equation (74b)] and the density function of the phase angle $\hat{\theta}$ is sharply peaked around $\pi/2$ [equation (73)]. Results have also been obtained for $\sigma_\tau = 0{\cdot}1$ in the present case (not plotted in Fig. 3) and these results checked very closely with the curves for $\sigma_\tau = 0{\cdot}1$ and $d = \infty$ in Fig. 2 for case (a). For these values of σ_τ and d, $\hat{\mathscr{E}}$ in both cases is essentially a random variable which is only slightly random.

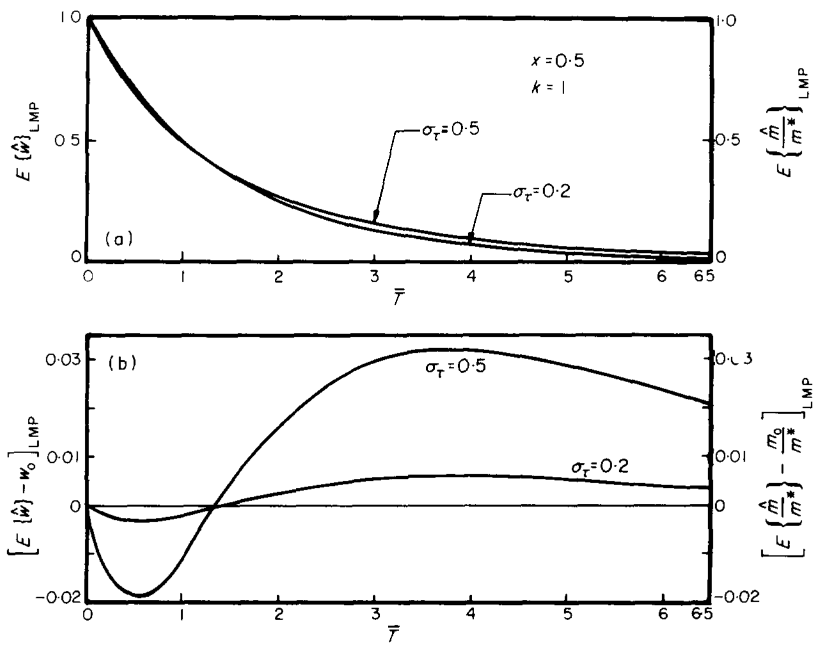

FIG. 3. Decay curves—case (b). (a) LMP of mean velocity and bending moment. (b) Deviation from LMP of deterministic velocity and bending moment.

The logarithmic decrement δ, a quantity widely used as a measure of damping, is given by the expression

$$\delta = \ln\left(\frac{W_1}{W_2}\right) \tag{79}$$

where W_1 and W_2 are the velocities at two successive maxima. It then follows from equation (72) that

$$\hat{\delta} = \frac{2\pi\hat{\mathscr{E}}}{\sqrt{[(\beta k^2/\alpha)^2 - \hat{\mathscr{E}}^2]}}. \tag{80}$$

Also, the circular frequency in cycles per second follows as

$$\hat{\omega} = \frac{\alpha\sqrt{[(\beta k^2/\alpha)^2 - \hat{\mathscr{E}}^2]}}{2\pi}. \tag{81}$$

For convenience, we rewrite equations (80) and (81) in nondimensional form as

$$\frac{\hat{\delta}}{\delta_0} = \frac{\hat{\mathscr{E}}\sqrt{[(\beta k^2/\alpha)^2 - 1]}}{\sqrt{[(\beta k^2/\alpha)^2 - \hat{\mathscr{E}}^2]}} \tag{82a}$$

$$\frac{\hat{\omega}}{\omega_0} = \frac{\sqrt{[(\beta k^2/\alpha)^2 - \hat{\mathscr{E}}^2]}}{\sqrt{[(\beta k^2/\alpha)^2 - 1]}} \tag{82b}$$

where δ_0 and ω_0 are the nominal values (at $\hat{T} = T_0$) given by

$$\delta_0 = \frac{2\pi}{\sqrt{[(\beta k^2/\alpha)^2 - 1]}} \qquad \omega_0 = \frac{\alpha\sqrt{[(\beta k^2/\alpha)^2 - 1]}}{2\pi}. \tag{83}$$

Given that $\hat{\mathscr{E}}$ is truncated lognormal [equation (74)], the density functions of $\hat{\delta}/\delta_0$ and $\hat{\omega}/\omega_0$ follow from equations (82) as

$$f\left(\frac{\delta}{\delta_0}\right) = \frac{Y_1^2[(\beta k^2/\alpha)^2-1]}{(\delta/\delta_0)\Omega_1\sigma_\tau\sqrt{(2\pi)}}\exp\left\{-\frac{[\ln(Y_1(\beta k^2/\alpha)(\delta/\delta_0))]^2}{2\sigma_\tau^2}\right\}U\left(\frac{\delta}{\delta_0}\right) \tag{84a}$$

$$Y_1 = \left[\left(\frac{\delta}{\delta_0}\right)^2+\left(\frac{\beta k^2}{\alpha}\right)^2-1\right]^{-\frac{1}{2}} \tag{84b}$$

and

$$f\left(\frac{\omega}{\omega_0}\right) = \frac{(\omega/\omega_0)[(\beta k^2/\alpha)^2-1]}{Z_1\Omega_1\sigma_\tau\sqrt{(2\pi)}}\exp\left\{-\frac{(\ln Z_1)^2}{8\sigma_\tau^2}\right\}\left[U\left(\frac{\omega}{\omega_0}\right)-U\left(\frac{\omega}{\omega_0}-\frac{\beta k^2/\alpha}{\sqrt{[(\beta k^2/\alpha)^2-1]}}\right)\right] \tag{85a}$$

$$Z_1 = \left(\frac{\beta k^2}{\alpha}\right)^2-\left(\frac{\omega}{\omega_0}\right)^2\left[\left(\frac{\beta k^2}{\alpha}\right)^2-1\right]. \tag{85b}$$

Equation (84a) has been plotted in Fig. 4 for the previously discussed example (solid curves) with $k = 1$ and typical values for σ_τ. The lower limit on the logarithmic decrement

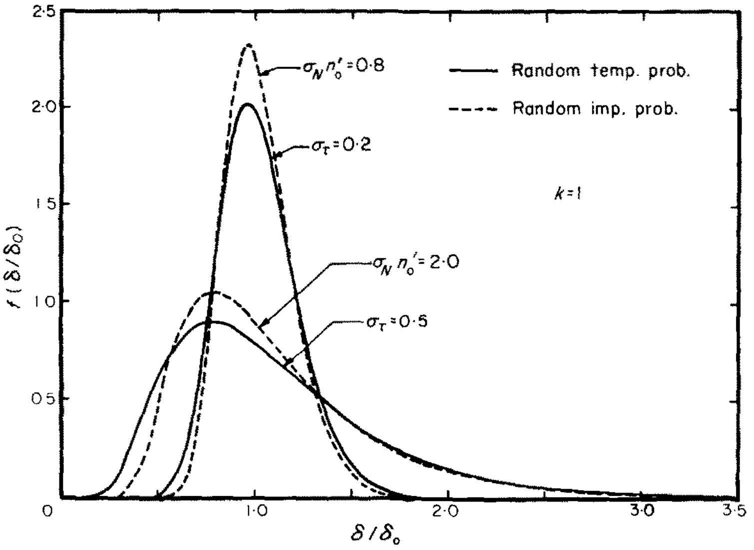

FIG. 4. Density function of $\hat{\delta}/\delta_0$—case (b).

($\delta/\delta_0 = 0$) corresponds with $\mathscr{E} = 0$, while the upper limit ($\delta/\delta_0 \to \infty$) corresponds with the cutoff $\mathscr{E} = \beta/\alpha$. Equation (85a) has also been plotted in Fig. 5 (solid curves) for the same values of k and σ_τ. Here, the lower limit on the circular frequency ($\omega/\omega_0 = 0$) corresponds with the cutoff $\mathscr{E} = \beta/\alpha$, while the upper cutoff ($\omega/\omega_0 = (\beta/\alpha)/\sqrt{[(\beta/\alpha)^2-1]}$) corresponds with $\mathscr{E} = 0$.

The means and the standard deviations of $\hat{\delta}/\delta_0$ and $\hat{\omega}/\omega_0$ are readily obtained by numerical integration. For σ_τ equal to 0·2 and 0·5, values of these statistical properties and those for $\hat{\mathscr{E}}$ [equations (11)] are given in Table 3.

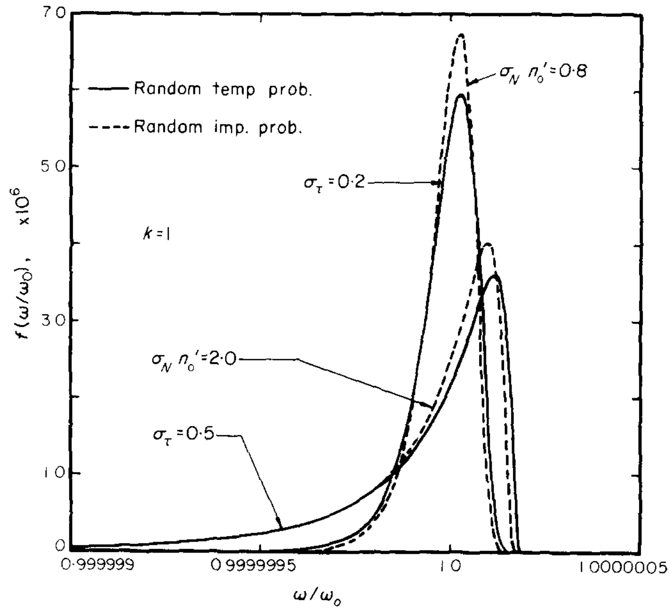

FIG. 5. Density function of $\hat{\omega}/\omega_0$—case (b).

TABLE 3. MEANS AND STANDARD DEVIATIONS OF $\hat{\delta}/\delta_0$ AND $\hat{\omega}/\omega_0$

σ_τ	$E\{\hat{\mathscr{E}}\}$	$\sigma_{\mathscr{E}}$	$E\{\hat{\delta}/\delta_0\}$	σ_{δ/δ_0}	$E\{\hat{\omega}/\omega_0\}$	σ_{ω/ω_0}
0·2	1·0202	0·2061	1·0202	0·2061	0·999999	0·0012
0·5	1·1331	0·6039	1·1331	0·6039	0·999590	0·0203

It may be seen from Table 3 that in this example the statistics of the logarithmic decrement are numerically equal to the statistics of the material parameter $\hat{\mathscr{E}}$. This follows directly from equation (82a) since the density function of $\hat{\mathscr{E}}$ peaks near value 1·0 and $\beta/\alpha \gg 1$ (small damping). Applying the same argument to equation (82b), we find that the density function of $\hat{\omega}/\omega_0$ is very sharply concentrated near 1·0, and this is apparent in Fig. 5 and Table 3. Thus, we find in this example that whereas the logarithmic decrement is sensitive to fluctuations in temperature, the circular frequency is almost insensitive to such fluctuations.

Bending moment. Since equations (69) and (70) with their associated initial and boundary conditions are mathematically similar, we may proceed in an analogous manner to obtain the bending moment and its statistics. We obtain for the bending moment

$$\hat{m}(x, t) = \frac{-(\beta k)^2}{2\pi^2\alpha^2\sqrt{[(\beta k^2/\alpha)^2-\hat{\mathscr{E}}^2]}}\,\mathrm{e}^{-\alpha\hat{\mathscr{E}}\bar{T}}\sin(\alpha\sqrt{[(\beta k^2/\alpha)^2-\hat{\mathscr{E}}^2]\bar{T}})\sin k\pi x \quad k = 1, 2, 3, \ldots \tag{86}$$

and for the mean bending moment

$$E\left\{\frac{\hat{m}}{m^*}\right\} = \frac{\beta k^2 \sin k\pi x}{\alpha\Omega_1\sqrt{(2\pi)}\sigma_\tau}\int_0^{\beta k^2/\alpha}\frac{\exp\left\{-\left(\alpha\mathscr{E}\bar{T}+\dfrac{(\ln\mathscr{E})^2}{2\sigma_\tau^2}\right)\right\}}{\mathscr{E}\sqrt{[(\beta k^2/\alpha)^2-\mathscr{E}^2]}}\sin(\alpha\sqrt{[(\beta k^2/\alpha)^2-\mathscr{E}^2]\bar{T}})\,\mathrm{d}\mathscr{E} \tag{87}$$

where m^* is defined in equation (57b).

The expression for $E\{\hat{m}/m^*\}$ given by equation (87) is identical with the expression for $E\{\hat{w}\}$ given by equation (78) except that here the phase angle $\hat{\theta}$ is equal to zero. Due to the high frequency in the previously discussed example we may expect that the locus of maximum points in that example is the same for bending moment and velocity. Thus, Fig. 3 also applies in this case. Likewise, equations (82), (84) and (85) for $\hat{\delta}/\delta_0$ and $\hat{\omega}/\omega_0$ and their density functions also hold for the bending moment solution. Thus the bending moment and its logarithmic decrement also show significant random fluctuations as the temperature fluctuates.

4.2. *Random imperfection problem*

Velocity. We now turn to the random imperfection problem in which only $\hat{n}$ is random. In this problem $\hat{w}$, $\hat{\delta}/\delta_0$ and $\hat{\omega}/\omega_0$ can be easily obtained from equations (72), (73), (82) and (83) in the random temperature problem with the substitution

$$\hat{\mathscr{E}} \to \hat{n}/n_0. \tag{88}$$

Now, δ_0 and ω_0 are nominal values evaluated at $\hat{N} = N_0$. The underdamped restriction in this problem requires that

$$\hat{n} < \frac{\beta k^2}{\alpha'} \tag{89a}$$

where

$$\alpha' = \alpha/n_0. \tag{89b}$$

Again, since there is zero probability that $\hat{n}$ in this case is greater than $\beta k^2/\alpha'$, the lognormal density function for $\hat{n}$ [equation (13)] must be replaced by a truncated lognormal density function. Thus, we write

$$f(n) = \frac{n_0 - 1}{\Omega_2\sqrt{(2\pi)}\sigma_N n_0'(n-1)} \exp\left\{-\frac{1}{2}\left(\frac{n_0-1}{\sigma_N n_0'}\right)^2\left[\ln\left(\frac{n-1}{n_0-1}\right)\right]^2\right\}\left[U(n-1) - U\left(n - \frac{\beta k^2}{\alpha'}\right)\right] \tag{90a}$$

$$\Omega_2 = \tfrac{1}{2} + \operatorname{erf}\left[\frac{n_0-1}{\sigma_N n_0'}\ln\left(\frac{(\beta k^2/\alpha') - 1}{n_0 - 1}\right)\right]. \tag{90b}$$

The mean $E\{\hat{n}\}$ and variance σ_n^2 are then given by

$$E\{\hat{n}\} = 1 + \frac{H_3}{\Omega_2}(n_0 - 1)\exp\left\{\frac{1}{2}\left(\frac{\sigma_N n_0'}{n_0 - 1}\right)^2\right\} \tag{91a}$$

$$\sigma_n^2 = \left(\frac{H_3}{\Omega_2}\right)^2 (n_0-1)^2 \exp\left\{\left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right\}\left[\frac{\Omega_2 H_4}{H_3^2}\exp\left\{\left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right\} - 1\right] \tag{91b}$$

where

$$H_3 = \tfrac{1}{2} + \operatorname{erf}\left\{\frac{n_0-1}{\sigma_N n_0'}\left[\ln\left(\frac{(\beta k^2/\alpha')-1}{n_0-1}\right) - \left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right]\right\} \tag{92a}$$

$$H_4 = \tfrac{1}{2} + \operatorname{erf}\left\{\frac{n_0-1}{\sigma_N n_0'}\left[\ln\left(\frac{(\beta k^2/\alpha')-1}{n_0-1}\right) - 2\left(\frac{\sigma_N n_0'}{n_0-1}\right)^2\right]\right\}. \tag{92b}$$

Given that $\hat{n}$ is truncated lognormal [equation (90a)], the density functions of $\hat{\delta}/\delta_0$ and $\hat{\omega}/\omega_0$ are obtained from equations (82) with substitution (88) as

$$f\left(\frac{\delta}{\delta_0}\right) = \frac{(n_0-1)\left(\frac{\beta k^2}{\alpha'}\right)\left[\left(\frac{\beta k^2}{\alpha'}\right)^2 - n_0^2\right]}{(n_0 Y_2)^2\left[\left(\frac{\beta k^2}{\alpha'}\right)^2\left(\frac{\delta}{\delta_0}\right) - Y_2\right]\Omega_2\sqrt{(2\pi)}\sigma_N n_0'}$$

$$\cdot \exp\left\{-\frac{1}{2}\left(\frac{n_0-1}{\sigma_N n_0'}\right)^2\left[\ln\left(\frac{\left(\frac{\beta k^2}{\alpha'}\right)^2\left(\frac{\delta}{\delta_0}\right) - Y_2}{(n_0-1)Y_2}\right)\right]^2\right\} U\left[\frac{\delta}{\delta_0} - \frac{1}{n_0}\sqrt{\left(\frac{(\beta k^2/\alpha')^2 - n_0^2}{(\beta k^2/\alpha')^2 - 1}\right)}\right] \quad (93a)$$

$$Y_2 = \left[\left(\frac{\delta}{\delta_0}\right)^2 + \frac{(\beta k^2/\alpha')^2 - n_0^2}{n_0^2}\right]^{\frac{1}{2}} \quad (93b)$$

$$f\left(\frac{\omega}{\omega_0}\right) = \frac{(n_0-1)\left(\frac{\omega}{\omega_0}\right)\left[\left(\frac{\beta k^2}{\alpha'}\right)^2 - n_0^2\right]}{Z_2(Z_2-1)\Omega_2\sqrt{(2\pi)}\sigma_N n_0'} \exp\left\{-\frac{1}{2}\left(\frac{n_0-1}{\sigma_N n_0'}\right)^2\left[\ln\left(\frac{Z_2-1}{n_0-1}\right)\right]^2\right\}$$

$$\cdot\left\{U\left(\frac{\omega}{\omega_0}\right) - U\left[\frac{\omega}{\omega_0} - \sqrt{\left(\frac{(\beta k^2/\alpha')^2 - 1}{(\beta k^2/\alpha')^2 - n_0^2}\right)}\right]\right\} \quad (94a)$$

$$Z_2 = \left\{\left(\frac{\beta k^2}{\alpha'}\right)^2 - \left(\frac{\omega}{\omega_0}\right)^2\left[\left(\frac{\beta k^2}{\alpha'}\right)^2 - n_0^2\right]\right\}^{\frac{1}{2}}. \quad (94b)$$

Equations (93a) and (94a) have been plotted in Figs. 4 and 5 respectively (dashed curves) for the previously discussed example with $k = 1$ and typical values for $\sigma_N n_0'$. The means and the standard deviations of $\hat{\delta}/\delta_0$ and $\hat{\omega}/\omega_0$ are also readily obtained. For $\sigma_N n_0' = 0{\cdot}8$ and $2{\cdot}0$, values of these statistical properties and those for $\hat{n}$ [equations (14)] are given in Table 4.

TABLE 4. MEANS AND STANDARD DEVIATIONS OF $\hat{\delta}/\delta_0$ AND $\hat{\omega}/\omega_0$

$\sigma_N n_0'$	$E\{\hat{n}/n_0\}$	σ_{n/n_0}	$E\{\hat{\delta}/\delta_0\}$	σ_{δ/δ_0}	$E\{\hat{\omega}/\omega_0\}$	σ_{ω/ω_0}
0·8	1·0201	0·1827	1·0201	0·1827	0·999938	0·0079
2·0	1·1342	0·5589	1·1342	0·5589	0·994273	0·0755

As in the random temperature problem we find that in this example the statistics of the logarithmic decrement are numerically equal to those of $\hat{n}/n_0$, and that the density function of $\hat{\omega}/\omega_0$ is very sharply concentrated near one. Thus, we again find that whereas the logarithmic decrement shows significant random fluctuation as the imperfection density fluctuates, the circular frequency is almost deterministic.

Bending moment. The bending moment for the random imperfection problem may be obtained from equation (86) with the substitution (88). We then find that the statistics of the bending moment are identical with those of the velocity, and thus require no further discussion.

5. SUMMARY OF RESULTS

The present study is an investigation of the effect of randomness in the material parameters $\hat{\varepsilon}_c$ and $\hat{n}$ on damped lateral vibration in an axially creeping beam. The randomness in these parameters results from randomness in the temperature and imperfection density, which have been assumed to be independent homogeneous normal random processes. Two separate cases have been considered, i.e., (a) $\hat{\mathscr{E}}$ and $\hat{n}$ random processes which are slightly random, and (b) $\hat{\mathscr{E}}$ and $\hat{n}$ random variables which are largely random. In each case the problem has been separated into two uncoupled parts—the "random temperature problem" and the "random imperfection problem". In the random temperature problem only $\hat{\varepsilon}_c$ is random, whereas in the random imperfection problem only n is random. For both cases a simply-supported nonlinear Maxwell beam made of aluminum alloy 1100 has been taken as an example. In this example the damping is small, and the vibration is underdamped.

In case (a) we have obtained the mean bending moment and velocity by using exponential-like autocovariances for the temperature and the imperfection density. As a consequence of the small damping, very simple and useful expressions for the mean bending moment and the mean velocity have been obtained. Two critical times, where the effect of material parameter randomness on the bending moment and velocity is an extremum, have been found. It has been observed that both the bending moment and velocity are sensitive to randomness in temperature and in imperfection density. This differs from the results obtained for the 3-bar truss in [5] and statically loaded beam in [6], where the stress was essentially deterministic and only the velocity was sensitive to such randomness.

In case (b) we have found the means of the velocity and the bending moment. Comparing these means with those in case (a), we have observed that the two cases give equivalent results when the correlation distance in case (a) is large and the standard deviation in case (b) is small. The statistics of the logarithmic decrements and circular frequencies for the velocity and bending moment have also been presented. It has been found that as a consequence of the small damping the velocity, bending moment and their associated logarithmic decrements are sensitive to randomness in temperature and in imperfection density, whereas the associated circular frequencies are almost deterministic.

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Абстракт—Исследуется затухающее поперечное колебание балки с осебой ползучестью и случайными параметрами материала. Температура и плотность неправильностей, и в результате, два неупругих параметра материала являются случайными функциями расстояния вдоль нейтральной оси. Предполагается материал балки в виде модели Максвелла, и также, что началальное напряжение растяжения значительно больше по сравнению с приростом напряжения вследствие колебания. Затем, можно применить метод возмущений. Исследуются полностью два специальных случая: /а/ случайные параметры являются случайными процессами, которые только слегка случайны, /б/ случайные параметры являются случайными переменными, которые в значительной мере случайны. Получаются статистические результаты для поперечной скорости, момента изгиба, логарифмического декремента и круговой частоты. Находится, что для специфического примера балки, изготовленной из алюминиевого сплава, поперечная скорость, момент изгиба и логарифмический декремент чувствительны к случайности температуры и плотности неправильностей, тогда как круговая частота почти детерминическа.